ELSEVIER

Discrete Applied Mathematics 120 (2002) 25–33

DISCRETE APPLIED MATHEMATICS

Triangle-free graphs and forbidden subgraphs

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Received 16 September 1999; received in revised form 5 April 2001; accepted 3 June 2001

Abstract

The relation of chromatic aspects and the existence of certain induced subgraphs of a triangle-free graph will be investigated. Based on a characterization statement of Pach, some results on the chromatic number of triangle-free graphs with certain forbidden induced subgraphs will be refined by describing their structure in terms of homomorphisms. In particular, we introduce chordal triangle-free graphs as a natural superclass of chordal bipartite graphs and describe the structure of the maximal triangle-free members. Finally, we improve on the upper bound for the chromatic number of triangle-free sK_2-free graphs by 1 for $s \geqslant 2$, giving the tight bound for $s=3$. © 2002 Published by Elsevier Science B.V.

A graph G is called perfect, if the chromatic number $\chi(G')$ equals the clique number $\omega(G')$ for every induced subgraph G' of G. According to the strong perfect graph conjecture due to Berge, perfect graphs are characterized by two infinite families of forbidden subgraphs: G is perfect if and only if G does not contain an odd cycle of length at least 5 nor its complement as an induced subgraph. This conjecture is still open. In 1973, Gyárfás [7] relaxed the concept of perfection. He called a class $\mathscr{G}$ of graphs χ-bound if there is a function f such that $\chi(G') \leqslant f(\omega(G'))$ for every induced subgraph G' of $G \in \mathscr{G}$. The function f is called a χ-binding function of $\mathscr{G}$ and we call a χ-binding function optimal if no smaller function is a χ-binding function for $\mathscr{G}$. In this paper, we concentrate on hereditary graph classes, where with the graph every induced subgraph belongs to the class. In this case it suffices to require that $\chi(G) \leqslant f(\omega(G))$ for the graph G itself.

If $\mathscr{G}$ is characterized by a finite family of forbidden induced subgraphs, then, in order to be χ-bound, one of the forbidden subgraphs must be a forest. This follows from a famous result of Erdős [6], showing that there are graphs with arbitrarily large girth and chromatic number. Gyárfás conjectured that the converse is true. For a forest F

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0166-218X/02/$ - see front matter
PII: S0166-218X(01)00277-3

let $\mathscr{G}(F)$ be the class of graphs that do not contain F as an induced subgraph. Gyárfás [7] conjectured that for every forest F the class $\mathscr{G}(F)$ is χ-bound.

For the two extreme families of trees, stars $K_{1,s-1}$ and paths P_s on s vertices, Gyárfás [8] proved that $\mathscr{G}(K_{1,s-1})$ and $\mathscr{G}(P_s)$ are indeed χ-bound. Kierstead and Penrice [10] extended the result for stars to trees of radius at most 2.

As far as optimal χ-binding functions are concerned, little is known. Since P_4-free graphs are perfect (Seinsche [17]), the optimal χ-binding function is $f_{P_4}(\omega)=\omega$. Certainly, for every induced subgraph H of P_4, the H-free graphs are perfect as well. The smallest forests that are not induced subgraphs of P_4 are $3K_1$ and $2K_2$. From a more general result of Gyárfás [8] follows that the optimal χ-binding function f_{3K_1} is essentially half of the Ramsey number $R(3,\omega+1)$, more precisely we get

$$\frac{R(3,\omega+1)-1}{2} \leqslant f_{3K_1}(\omega) \leqslant f_{K_1\cup P_3}(\omega) \leqslant \frac{R(3,\omega+1)-1+\omega}{2}. \tag{1}$$

Using Kim's famous result [11] the upper and lower bounds are both $\Theta(\omega^2/\log\omega)$. The dependence on the Ramsey number indicates that the exact calculation of $f_{3K_1}(\omega)$ is currently out of reach for larger ω. For $f_{2K_2}(\omega)$ we have $\omega^{1+\epsilon} \leqslant f_{2K_2}(\omega) \leqslant \binom{\omega+1}{2} = \mathcal{O}(\omega^2)$ for a small $\epsilon>0$ due to Gyárfás [8] (lower bound) and Wagon [19] (upper bound), so even the precise asymptotics is open.

In order to obtain further exact values of optimal χ-binding functions we have to restrict to graphs with small clique number. Even for $\omega=3$ not much more seems to be known. For every induced subgraph H of the so-called chair, the graph obtained from the star $K_{1,3}$ by subdividing an edge once, Randerath [14] proved that $f_H(3)=4$, if H is not an induced subgraph of P_4. By an unpublished result of Nagy and Szentmiklóssy answering a \$20 problem of Erdős (see [8]) we have $f_{2K_2}(3)=4$ as well.

Let us now turn to the case $\omega=2$ which we will investigate in more detail. A *homomorphism* of a graph G with a graph H is an edge-preserving mapping $\sigma: V(G)\to V(H)$. Homomorphisms are a refinement of graph colouring since G is k-colourable if and only if it is homomorphic with K_k. It is not difficult to see that every graph has a unique minimal induced subgraph with which it is homomorphic, the *core* of the graph (see [9]). A triangle-free graph is *maximal* triangle-free, if joining any two non-adjacent vertices creates a triangle. If we have a homomorphism from a maximal triangle-free graph to a triangle-free graph then the only vertices that can be identified by the homomorphism are those having exactly the same neighbourhood which we will call similar (they have also been called symmetric or twins). This implies that the core of a maximal triangle-free graph is obtained by successively deleting vertices having the same neighbourhood as a remaining vertex until no such pair of vertices is left.

In our investigations, a specific class of 3-colourable maximal triangle-free graphs, denoted Γ_i, will play a central role. The vertices of Γ_i are the corners of a regular $(3i-1)$-gon inscribed in a circle and two vertices are adjacent if they are further apart than the corners of a regular inscribed 3-gon (the graphs Γ_i in a slightly different, unscrambled representation are indicated in Fig. 1). This sequence probably first appeared in Andrásfai [1] attributing the construction to Erdős and himself, and was

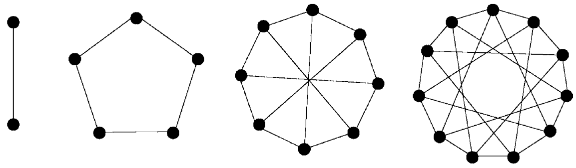

Fig. 1. The graphs Γ_i, $1 \leqslant i \leqslant 4$.

rediscovered many times over the years. The graphs Γ_i turn out to have induced cycles of only a few lengths.

Lemma 1. *Let C be an induced cycle of a graph Γ_i. Then $4 \leqslant |C| \leqslant 5$ or $i \geqslant 5$ and $|C| = 7$.*

Proof. Let C be an induced cycle of Γ_i. Fix an edge $v_1v_2 \in E(C)$ and consider the path P of C induced by the edges which have no endvertex in $N_C(v_1) \cup N_C(v_2)$. Now consider the neighbourhoods $N_{\Gamma_i}(v_1)$ and $N_{\Gamma_i}(v_2)$. They are disjoint sets of consecutive vertices on the circle, so the remaining vertices form two intervals of the circle. The vertices within an interval are not adjacent because both intervals together cover less than one-third of the circle. Any two vertices from different intervals are adjacent, since the distance is bigger than a vertex neighbourhood. So the vertices of P form an induced complete bipartite subgraph of Γ_i, which implies that P has at most 3 vertices, so $|C| \leqslant 7$.

If C contains two geometrically disjoint edges v_1w_1, v_2w_2 then the two diagonals v_1w_2 and v_2w_1 are both longer than the shortest edge of v_1w_1, v_2w_2 and hence $|C| = 4$. In particular, every two edges of an induced cycle of length more than 4 intersect.

Assume that C is an even cycle of length $|C| > 4$, where every two edges intersect. Fix an edge v_1v_2 and consider the path P spanned by the edges of C that are not incident with v_1, v_2. The path P has odd length and since all edges of P cross the edge v_1v_2, the endvertices of P are separated by v_1v_2. So the two edges joining v_1v_2 and P in C are separated by v_1v_2 as well and cannot intersect. So every induced cycle of length more than 4 is odd.

Finally, assume that Γ_4 contains an induced 7-cycle C. Since Γ_4 is 4-regular of order 11, $7 + 14 = 21$ of its edges are incident with a vertex of C. So there is a vertex in $G - C$, having all its neighbours in C, which is impossible in a 4-regular triangle-free graph. Since every graph Γ_i, $i \leqslant 4$, is an induced subgraph of Γ_4, it cannot contain an induced 7-cycle. Note that choosing every second vertex of Γ_5 yields an induced 7-cycle, and since every graph Γ_i, $i \geqslant 5$, contains Γ_5 as an induced subgraph, it contains an induced 7-cycle from Γ_5.

The following characterization statement due to Pach [13] turns out to be the central tool in many of the following arguments. In a triangle-free graph, the neighbourhood of every vertex forms an independent vertex set. Pach [13] characterized those triangle-free graphs where the converse — every independent set is contained in the neighbourhood of a vertex — holds as well. These are exactly those graphs which are maximal triangle-free and homomorphic with a graph Γ_i. The result was recently rediscovered by Brouwer [4] with a much simpler proof. The following extension of Pach's result was observed in [3]. The heart of the proof presented here is a variation of Brouwer's proof.

Theorem 2. *Let G be a triangle-free graph with at least two vertices. Then the following four statements are equivalent*:
(1) *every independent vertex set is contained in the neighbourhood of a vertex*,
(2) *G is maximal triangle-free and homomorphic with Γ_i for some $i \geqslant 1$*,
(3) *G is maximal triangle-free and does not contain an induced* 6-*cycle*, *and*
(4) *G is maximal triangle-free and does not contain the vertex deleted Petersen graph as a* (*not necessarily induced*) *subgraph.*

Proof. Let us first show the equivalence of (3) and (4). Let G be a maximal triangle-free graph. If G contains an induced C_6 then each of the three pairs of antipodal vertices on C_6 must have a common neighbour. No two pairs can have the same neighbour so the graph induced by C_6 and the three common neighbours contains $H_{10}-v$. Conversely, if G contains $H_{10}-v$ as a subgraph, then the graph induced by the degree 3 vertices of $H_{10}-v$ is an induced C_6.

The implication (2) $\Rightarrow$ (3) immediately follows from the fact, that no graph Γ_i contains an induced cycle other than C_4, C_5, C_7. So let us show that (3) $\Rightarrow$ (1). Let G be a maximal triangle-free graph, containing an independent vertex set U, which is not contained in the neighbourhood of a vertex. Then U has a minimal subset $U' = \{u_1, u_2, \dots, u_k\}$, which is also not contained in the neighbourhood of a vertex. By the maximality of G, $|U'| \geqslant 3$. By the minimality of U', for every vertex u_i there is a vertex w_i which is adjacent to every vertex in U' except u_i. Now the graph induced by the vertex set $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ is C_6.

We will now complete the proof by proving (1) $\Rightarrow$ (2) inspired by Brouwer's proof. Let G have property (1). Since any independent set of two vertices has a common neighbour, G has diameter 2 and is therefore maximal triangle-free. Consider the core G' of G. Since the core G' is obtained from G by deleting vertices with the same neighbourhood as a remaining vertex, G' satisfies (1) as well. For every ordered pair of non-adjacent vertices (v, w) of G' there is a unique neighbour $\phi(v, w)$ of v which is adjacent to all vertices of $N(w) \setminus N(v)$. Since $\{v\} \cup (N(w) \setminus N(v))$ is an independent set such a vertex exists. Assuming that there are two vertices x_1, x_2 with this property then there is a neighbour y_1 of x_1 which is not adjacent to x_2 since G' a core. Now y_1, x_2, w form an independent set which is not contained in the neighbourhood of a vertex, a contradiction.

We will proceed by induction on the order of G'. If $|G'| \leqslant 2$ then $G' = K_2 = \Gamma_1$, so assume that $|G'| \geqslant 3$ and hence G' contains two non-adjacent vertices. Let v_1, v_2 be a pair of non-adjacent vertices for which the intersection $|N(v_1) \cap N(v_2)|$ is as small as possible. We will show that $N(v_1) \cap N(v_2)$ contains a single vertex v_3. Assume otherwise and consider $u = \phi(v_1, v_2)$ and $x \in N(v_1) \cap N(v_2)$. Since $|N(u) \cap N(x)| \geqslant 2$ there must be a vertex $u' \in N(u) \setminus N(v_2)$, and since G' is a core there is a vertex in $w \in N(v_1) \setminus N(u')$. If $w \in N(v_2)$ then (u', v_2) is a pair of non-adjacent vertices with $|N(u') \cap N(v_2)| < |N(v_1) \cap N(v_2)|$, a contradiction. Otherwise u', v_2, w is an independent set of vertices that contradicts (1).

Consider the graph $G'' = G' - \{v_1, v_2, v_3\}$. First note that every vertex of G'' has exactly one neighbour in $\{v_1, v_2, v_3\}$. Indeed, since v_1 and v_2 have exactly one common neighbour v_3, every vertex of G'' has at most one neighbour in $\{v_1, v_2, v_3\}$. Since for every vertex x which is neither adjacent to v_1 nor v_2 the set $\{v_1, v_2, x\}$ is independent and has a common neighbour which must be v_3, every vertex has at least one neighbour in $\{v_1, v_2, v_3\}$. Next, observe that for every i, $1 \leqslant i \leqslant 3$, there is a vertex $v_i' \in V(G'')$, such that $N_{G''}(v_i') \supseteq N_{G''}(v_i)$, with equality if $i \in \{1, 2\}$. Indeed, $v_1' = \phi(v_2, v_1)$ and $v_2' = \phi(v_1, v_2)$ since for $\{i, j\} = \{1, 2\}$ we have $N_{G''}(v_i) \subseteq N(\phi(v_j, v_i))$ and a neighbour $x \neq v_j$ in $N(\phi(v_j, v_i)) \setminus N(v_i)$ would have a non-neighbour $y \in N(v_j)$ different from v_3, such that $\{v_i, x, y\}$ would be an independent set without a common neighbour. Finally, $v_3' = \phi(v_1', v_3)$ has the indicated property.

We will now show that G'' satisfies (1) and is a core. Assume that G'' has an independent set S that is not contained in the neighbourhood of a vertex. Since S was in G' in the neighbourhood of a vertex it must properly be contained in the neighbourhood of a vertex v_i. But then S is contained in the neighbourhood of $v_i' \in V(G'')$. In particular, G'' is maximal triangle-free as well. Now assume that G'' is not a core which implies that there are two vertices $x, y \in V(G'')$ such that $N_{G''}(x) = N_{G''}(y)$ using that G'' is maximal triangle-free. Consider $x' = \phi(x, y)$ and $y' = \phi(y, x)$ which must belong to $\{v_1, v_2, v_3\}$, so $x'y' \in E(G')$ and by symmetry we may assume that $x' = v_1$ and $y' = v_3$. But now each vertex of $N(x) \cap N(y)$ is adjacent to v_2 and hence $N(y) \subseteq N(v_2)$, a contradiction to G' being a core.

Since G'' is a core which satisfies (1), we know by induction that $G'' = \Gamma_j$ for some $j \geqslant 1$. Since $N_{G''}(v_1)$ and $N_{G''}(v_2)$ are neighbourhoods of vertices of G'', they must form intervals of consecutive vertices on the circle. Moreover, since every vertex of G'' has exactly one neighbour v_i, the intervals must be disjoint, and since G' is triangle-free the neighbours $N_{G''}(v_3)$ also form an interval of consecutive vertices on the circle (and not two). Since $G'' = \Gamma_j$ is vertex transitive with cyclic rotation, the resulting graph G' is uniquely determined up to isomorphism and $G' = \Gamma_{j+1}$.

The equivalence of (1) and (2) is Pach's original statement. The equivalence of (2) and (3) will be the main tool in the present investigations. The equivalence of (1), (2) and (4) has been used in [3] to describe the structure of triangle-free graphs with large minimum degree in more detail.

A graph is called (k, ℓ)-*chordal* if every cycle of length at least k has at least ℓ chords (cf. [2, Chapter 3] for more information on (k, ℓ)-chordal graphs). The $(4, 1)$-chordal graphs are the usual chordal graphs. Restricting to triangle-free graphs, the $(4, 1)$-chordal graphs are just the acyclic graphs (forests). The $(5, 1)$-chordal triangle-free graphs form the well-studied family of so-called chordal bipartite graphs.

They are exactly the bipartite members of the $(6,1)$-chordal triangle-free graphs, using the fact that every non-bipartite graph has an induced odd cycle. Note that every cycle of length 4 or 5 in a triangle-free graph is necessarily induced. So the $(6,1)$-chordal triangle-free graphs are those triangle-free graphs for which every cycle that must not be induced by its length has a chord. In analogy to chordal bipartite graphs, let us call those graphs *chordal triangle-free*. Recently, Randerath and Schiermeyer [15] proved that every triangle-free graph which has induced cycles of only one even and one odd length is 3-colourable, implying that chordal triangle-free graphs are 3-colourable. We start by proving that every maximal triangle-free, chordal triangle-free graph has a much more restricted structure.

Theorem 3. *A maximal triangle-free graph is chordal triangle-free if and only if it is homomorphic with* Γ_4.

Proof. Let G be a maximal triangle-free graph. By Theorem 2, G contains an induced 6-cycle, and is therefore not chordal triangle-free, unless G is homomorphic with a graph Γ_i. If G is homomorphic with a graph Γ_i, but not homomorphic with Γ_4, then G contains Γ_5 as an induced subgraph. Since Γ_5 contains an induced 7-cycle by Lemma 1, G cannot be chordal triangle-free.

Conversely, suppose that G is homomorphic with Γ_4 and let C be an induced cycle of G of length more than 4. Then C is an induced subgraph of Γ_4 as well, since no two vertices of C have the same neighbourhood. Now, by Lemma 1, we get that $|C|=5$, so G is chordal triangle-free.

Now turn to the case of forbidden paths P_k of small order k. Here Gyárfás [8] offers an upper bound for the optimal χ-binding function of $f_{P_k}(2) \leqslant k-1$ for the triangle-free case. Since every P_4-free graph is perfect, triangle-free P_4-free graphs must be bipartite. What is the maximal k such that every triangle-free P_k-free graph is 3-colourable? Soon we will see that the answer is 5. Sumner [18] proved that 3-colourablility holds for a superclass of the P_5-free triangle-free graphs, namely those that are P_6-free and C_6-free. Note that these graphs are chordal triangle-free since an induced cycle of length more than 6 contains an induced P_6. So the above mentioned result of Randerath and Schiermeyer [15] implies 3-colourability as well. We will now show that the structure of these graphs is much more restricted. At the same time, our proof based on Theorem 3 is simpler than Sumner's proof of the weaker result.

Theorem 4. *The core of a triangle-free*, C_6*-free*, *and* P_6*-free graph is* K_1 *or* Γ_i, $1 \leqslant i \leqslant 3$.

Proof. Let G be a connected component of a triangle-free, C_6-free and P_6-free graph. Continue deleting a vertex v satisfying $N(v) \subseteq N(u)$ for a vertex $u \neq v$, until we end with a graph G', where no pair of vertices has the indicated property. Obviously, by the definition of G' we get that G is homomorphic with G'. We will now show that G'

has diameter at most 2. Suppose, to the contrary, that G' has two vertices v_1 and v_4 at distance 3 and let v_1, v_2, v_3, v_4 be a path of length 3 joining them. Since $N(v_1) \nsubseteq N(v_3)$ there must be a neighbour v_0 of v_1 that is not adjacent to v_3. Since G is triangle-free, v_0 is not adjacent to v_2 and v_0 is not adjacent to v_4, since the distance of v_1 and v_4 is 3. By an analogous reasoning we get a neighbour v_5 of v_4 that is not adjacent to v_1, v_2, v_3. Now if $v_0v_5 \in E(G)$ then $v_0, v_1, v_2, v_3, v_4, v_5$ is an induced C_6, otherwise it forms an induced P_6 in G' and, since G' is an induced subgraph of G, in G as well, a contradiction.

So we may assume that G' has diameter at most 2, i.e. G' is maximal triangle-free. Since G' is P_6-free, it cannot contain an induced cycle of length more than 6, and since G' is C_6-free we get that G' is chordal triangle-free. By Theorem 3 we get that G' is homomorphic with Γ_4. By construction, G' must be an induced subgraph of G. Since it is easy to see, that Γ_4 contains an induced P_6 we get that G' is a proper subgraph of Γ_4. Now it is easy to check that the only proper induced subgraphs of Γ_4 which are maximal triangle-free are K_1 and Γ_i, $1 \leqslant i \leqslant 3$. So G' is isomorphic with one of the indicated graphs and, since the indicated graphs are cores, G' is the core of G.

Finally, let us return to our original graph. Every component G has K_1 or Γ_i, $1 \leqslant i \leqslant 3$, as its core. Choose the largest order core of the components. This is a core of the whole graph since the cores of all components are homomorphic with this core.

Since the homomorphism relation is transitive and the core is 3-colourable, the graph is 3-colourable as well. Note that Liu and Zhou [12] characterized the triangle-free members of the P_6-free and C_6-free graphs as those triangle-free graphs, for which every connected induced subgraph has a dominating complete bipartite subgraph. Nevertheless, for our purposes this strong result does not seem to be of much help.

The Grötzsch graph is triangle-free, P_6-free and 4-chromatic. Very recently, Randerath et al. [16] have shown, that every connected triangle-free, P_6-free graph without similar vertices and with chromatic number at least 4 contains the Grötzsch graph as an induced subgraph and is an induced subgraph of the Clebsch graph and therefore 4-colourable. Also in this case there is a characterization result of Liu and Zhou [12]: A triangle-free graph is P_6-free if and only if every connected induced subgraph has a dominating complete bipartite subgraph or a dominating C_6.

Graphs that are P_6- and C_6-free form a superclass of the P_5-free graphs. Randerath [14, Lemma 33] characterized the connected triangle-free P_5-free graphs that are not bipartite, slightly refining a result of Sumner [18].

Theorem 5 (Randerath [14]). *A connected triangle-free, P_5-free graph is bipartite, or it is maximal triangle-free, and homomorphic with C_5.*

Without much work, this result can be derived from Theorem 4, but the direct proof is also not difficult. In the case of K_4-free graphs, the maximal chromatic number of a

K_4-free, P_5-free graph is apparently not known. A construction due to Randerath [14] shows that 5 is a lower bound. On the other hand, it can be easily derived from a characterization of Liu and Zhou [12] that every K_4-free, P_5-free graph has a connected dominating subgraph on at most 3 vertices, giving rise to an upper bound of 9 in view of Theorem 5. So $5 \leqslant f_{P_5}(3) \leqslant 9$, but we believe that the lower bound might be the truth.

Finally, let us return to sK_2-free graphs, i.e. graphs that do not contain an induced matching with s edges. We use a simple refinement of Wagon's result for $s = 2$ due to Chung et al. [5, Theorem 2], to improve on the case $s \geqslant 3$ for triangle-free graphs. Note that the statement is very similar to Theorem 5, except that the connectivity requirement can be dropped.

Theorem 6 (Chung et al. [5])**.** *A triangle-free, $2K_2$-free graph without isolated vertices is bipartite, or it is maximal triangle-free, and homomorphic with C_5.*

It is easy to see that

$$f_{sK_2}(2) \leqslant f_{(s-1)K_2}(2) + 2, \tag{2}$$

giving the upper bound $f_{sK_2}(2) \leqslant 2s - 1$ [19]. Based on the more refined description of Theorem 6 we can reduce this bound by 1 if $s \geqslant 3$.

Theorem 7. *Every triangle-free, $3K_2$-free graph is 4-colourable.*

Proof. It suffices to prove this result for every connected component G of the graph. Assume, to the contrary, that G is a connected triangle-free, $3K_2$-free graph which is not 4-colourable. For an edge vw, let $G(vw)$ be the subgraph induced by $V(G) \setminus (N(v) \cup N(w))$. Note that for every edge vw of G, the graph $G(vw)$ is $2K_2$-free. Moreover, $G(vw)$ is not bipartite, otherwise for a bipartition (X, Y) of $G(v, w)$, the sets $N(v), N(w), X, Y$ are the colour classes of a 4-colouring of G. By Theorem 6, $G(vw)$ induces a non-bipartite maximal triangle-free graph which is homomorphic with C_5. In particular, each edge of $G(vw)$ belongs to a 5-cycle C of $G(vw)$, and each vertex of $G(vw)$ has two neighbours on C.

Choose an induced matching of two edges u_1u_2, w_1w_2, which must exist since the graph is not 4-colourable. Partition $V(G)$ into three sets V_i, $0 \leqslant i \leqslant 2$, consisting of the vertices with exactly i neighbours in $\{u_1, u_2, w_1, w_2\}$. Since G is $3K_2$-free, V_0 is an independent set of vertices. Note that the edge w_1w_2 belongs to a 5-cycle $C_w = (w_1, w_2, w_3, w_4, w_5)$ (labeled in cyclic order) in $G(u_1u_2)$, so $w_3, w_5 \in V_1$ and $w_4 \in V_0$. Since V_0 is an independent set in $G(u_1u_2)$, every vertex of V_0 is adjacent to w_3 and w_5, since it must be adjacent to two vertices on C_w. Analogously, there must be a 5-cycle $C_u = (u_1, u_2, u_3, u_4, u_5)$ through u_1u_2 in $G(w_1w_2)$ with the same properties, in particular, every vertex of V_0 being adjacent to u_3 and u_5. So $\{u_3, u_5, w_3, w_5\}$ form an independent set.

Finally, consider $G(u_1u_5)$. By the maximality of $G(u_1u_5)$, w_3 and w_5 have a common neighbour w_4' in $G(u_1u_5)$, creating a 5-cycle $C_{w'} = (w_1, w_2, w_3, w_4', w_5)$ in $G(u_1u_5)$. But now u_3 is in $G(u_1u_5)$, but has at most one neighbour (namely w_4') in $C_{w'}$, a contradiction to $G(u_1u_5)$ being $2K_2$-free and hence to G being $3K_2$-free.

As already observed by Wagon [19], this result is best possible in view of the Grötzsch graph, which is triangle-free, 4-chromatic and $3K_2$-free — in fact there are further structurally different examples. Combining Theorem 7 with (2) we immediately get the following result.

Corollary 8. *Every triangle-free, sK_2-free graph ($s \geqslant 3$) is $(2s-2)$-colourable.*

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